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$O(5)\times U(1)$ Electroweak Gauge Theory and the Neutrino Oscillations in Matter

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A study is made of the group $O(5)\times U(1)$. The group is economical in the number of gauge bosons, which we associate with each of its generators, and is anomaly-free. The left-handed leptons $L_L^T \equiv (\nu_e, e, \mu, \nu_\mu)_L$ are assigned to the four-dimensional spinorial representations of $O(5)$. The right-handed particles are taken to be the singlets of the group. The theory has three sets of gauge bosons: (1) analogues of the GWS model, (2) additional charged gauge bosons, and (3) a set of three additional neutral gauge bosons as compared to the GWS model. We introduce neutrino mixing by mixing the additional charged gauge bosons. We develope a theory of neutrino oscillations in matter in such a way that in the absence of matter the scattering length reduces to the usual scattering length in vacuum. Even if the neutrino masses are equal or the neutrinos are massless, we still have neutrino oscillations in matter, a result already noted by Wolfenstein.

1. INTRODUCTION

Many features of the weak interaction phenomena can be explained by the $SU(2)\times U(1)$ gauge model of weak and electromagnetic interactions. This model was originally given by Glashow (1961), Weinberg (1967), and Salam (1968) (GWS). It was later extended to include four quark and four lepton flavors, which satisfy the Glashow-Iliopoulos-Maini (GIM) (Glashow *et al.*, 1970) mechanism. Many extensions of the GWS theory have been introduced for certain specific purposes (e.g., Langaker, 1981). One possible way of extending the theory is to employ larger groups, which increases the number of gauge bosons compared to the original model. We have used the group $O(5)\times U(1)$. The group $O(5)$ is anomaly-free and economical in the number of gauge bosons, which, in our work we associate with each of its generators.

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In Section 2 we sketch the relevant mathematics and in Section 3, besides giving the table of quantum numbers and defining the charge operator, we introduce neutrino mixing by mixing the additional charged gauge bosons. In Section 4 we summarize the usual procedure for obtaining neutrino oscillations in vacuum. Section 5 deals with neutrino oscillations in matter. Section 6 discusses the results.

2. THE RELEVANT MATHEMATICS

By using the method of Brauer and Weyl (1935) of constructing the spinorial representations for higher dimensional groups, we construct the four-dimensional spinorial representations of the five-dimensional rotation group $O(5)$. The set of five 4×4 hermitian anticommuting matrices Γ_a is made to satisfy the the following relation:

$$\Gamma_a^\dagger = \Gamma_a, \qquad \{\Gamma_a, \Gamma_b\} = 2i\delta_{ab} I, \qquad \text{where } a, b = 1, \dots, 5 \tag{1}$$

I is 4×4 unit matrix, and

$$\begin{gathered}\Gamma_1 = \sigma_1^{(1)} \times \sigma_1^{(2)}, \qquad \Gamma_2 = \sigma_1^{(1)} \times \sigma_2^{(2)}, \qquad \Gamma_3 = \sigma_3^{(1)} \times I \\ \Gamma_4 = \sigma_1^{(1)} \times \sigma_3^{(2)}, \qquad \Gamma_5 = \sigma_2^{(1)} \times I\end{gathered} \tag{2}$$

The superscripts (1) and (2) refer to two distinct sets of Pauli matrices and the symbol $\times$ stands for the direct product. The generators are given by

$$F_{ab} = -\tfrac{1}{2} i \Gamma_a \Gamma_b, \qquad a \neq b \tag{3}$$

The restriction is imposed due to the antisymmetry of F_{ab}. Explicitly written out, the matrices read

$$\begin{gathered}\Gamma_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \qquad \Gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \Gamma_4 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \qquad \Gamma_5 = \begin{pmatrix} 0 & -i\times 1 \\ i\times 1 & 0 \end{pmatrix}\end{gathered} \tag{4}$$

and the generators are given as follows:

$$\begin{gathered}F_{12} = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad F_{13} = \frac{1}{2}\begin{pmatrix} 0 & i\sigma_1 \\ -i\sigma_1 & 0 \end{pmatrix}, \qquad F_{14} = \frac{1}{2}\begin{pmatrix} -\sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix} \\ F_{15} = \frac{1}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \qquad F_{23} = \frac{1}{2}\begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, \qquad F_{24} = \frac{1}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \\ F_{25} = \frac{1}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \qquad F_{34} = \frac{1}{2}\begin{pmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix}, \qquad F_{35} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \\ F_{45} = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}\end{gathered} \tag{5}$$

The generators satisfy the following commutation relations:

$$[F_{ab}, F_{cd}] = i(\delta_{ac}F_{bd} - \delta_{bc}F_{ad} + \delta_{bd}F_{ac} - \delta_{ad}F_{bc}) \tag{6}$$

forming the corresponding Lie algebra.

For our purposes it is convenient to consider the algebra in a different basis,

$$\{F_i, F_{45}, F_i^{\pm}\}, \qquad i = 1, 2, 3 \tag{7}$$

defined by

$$F_1 = F_{23}, \qquad F_2 = F_{13}, \qquad F_3 = F_{12} \tag{8}$$

$$F_1^{\pm} = F_{14} \pm iF_{15}, \qquad F_2^{\pm} = F_{24} \pm iF_{25}, \qquad F_3^{\pm} = F_{34} \pm iF_{35} \tag{9}$$

Among the above set of generators using equation (6), in particular, the following commutation relations can be established:

$$[F_i^{\pm}, F_i^{\mp}] = \pm 2F_{45} \qquad (i \text{ not summed})$$

$$[F_{45}, F_i^{\pm}] = \pm F_i^{\pm} \tag{10}$$

$$[F_{45}, F_i] = 0 \tag{11}$$

$$[F_i, F_j] = i\varepsilon_{ijk}F_k \tag{12}$$

From equation (10) we see that for every value of $i\,(=1, 2, 3)$ the set of generators $\{F_{45}, F_i^{\pm}\}$ and from equation (12), the other one, i.e., $\{F_i\}$, form $su(2)$ subalgebras. Since the charge operator equation (13) is defined using the generator F_{45}, equations (10) and (11) indicate that the generators F_i and $F_i^{\pm}$ are the eigenvectors of the charge operator with the eigenvalues 0 and ± 1, and that the charge is invariant under the group $O(5)$ and eventually under the larger group $O(5)\times U(1)$.

3. THE $O(5)\times U(1)$ MODEL

This model has ten gauge fields W_{ij} $(i<j=1,\ldots,5)$ transforming as $O(5)$ generators and a singlet vector gauge field W_μ^0. We assign the left-handed leptons to the four-dimensional spinorial representation of $O(5)$ and denote these multiplets by $L_L^T = (\nu_e, e, \mu, \nu_\mu)_L$. The right-handed particles are taken to be the singlets of the group. We give the eigenvalues of the operators F_{45} and F_0 for the leptons in Table I.

Table I. Leptons

	$\nu_{eL,R}$	$e_{L,R}$	$\mu_{L,R}$	$\nu_{\mu L,R}$
Q	0	-1	-1	0
Y_{45}	$\frac{1}{2}, 0$	$-\frac{1}{2}, 0$	$-\frac{1}{2}, 0$	$\frac{1}{2}, 0$
Y_0	$-1, 0$	$-1, -2$	$-1, -2$	$-1, 0$

The eigenvalue of Y_{45} of the operator F_{45} is taken to be zero for the right-handed particles, as they are singlets and do not belong to the four-dimensional representation of $O(5)$. In terms of $O(5)$ and $U(1)$ generators the charge operator is given as

$$Q = F_{45} + \tfrac{1}{2}F_0 \tag{13}$$

It is possible to define a basis for the gauge bosons such that in the Lagrangian equation (19) certain linear combinations of the gauge field, for example, $(1/\sqrt{2})(W_\mu^{24} \pm iW_\mu^{25})$, can be universally coupled to the charged currents $\bar{\nu}_{eL}\gamma^\mu\frac{1}{2}(1+\gamma^5)e_L$ and $\bar{\nu}_{\mu L}\gamma^\mu\frac{1}{2}(1+\gamma^5)\mu_L$ rather than the separate ones W_μ^{24} and W_μ^{25}. We define

$$\begin{aligned} &F_C = F_{12}, \qquad F_D = F_{13}, \qquad F_E = F_{23}, \qquad F_F = F_{45} \\ &F_U^\pm = \frac{1}{\sqrt{2}}F_1^\pm, \qquad F_V^\pm = \frac{1}{\sqrt{2}}F_2^\pm, \qquad F_W^\pm = \frac{1}{\sqrt{2}}F_3^\pm \end{aligned} \tag{14}$$

and the corresponding basis for the gauge particles is taken as

$$\begin{aligned} &C_\mu = W_\mu^{12}, \qquad D_\mu = W_\mu^{13}, \qquad E_\mu = W_\mu^{23}, \qquad F_\mu = W_\mu^{45} \\ &U_\mu^\pm = \pm\frac{i}{\sqrt{2}}(W_\mu^{14} \mp iW_\mu^{15}), \qquad V_\mu^\pm = \mp\frac{i}{\sqrt{2}}(W_\mu^{34} \mp iW_\mu^{35}) \\ &W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^{24} \mp iW_\mu^{25}) \end{aligned} \tag{15}$$

Denoting the gauge couplings for the groups $O(5)$ by g and for $U(1)$ by $\frac{1}{2}g'$, we can express the couplings of the fermion currents to the gauge bosons (with $\bar{a}$ defined as the Dirac conjugate of a) and the abbreviations

$$a_L \equiv \tfrac{1}{2}(1+\gamma^5)a, \qquad a_R \equiv \tfrac{1}{2}(1-\gamma^5)a$$

$$\bar{a}\gamma^\mu b \to \bar{a}\gamma^\mu\tfrac{1}{2}(1+\gamma^5)b$$

by the following interaction Lagrangian

$$\begin{aligned} L_{\text{int}} = g &\sum_{i<j} (\bar{\psi}_L\gamma^\mu F_{ij}W^{ij}\psi_L) \\ &-\tfrac{1}{2}g'[(\bar{\nu}_{eL}\gamma^\mu\nu_{eL} + \bar{e}_L\gamma^\mu W_\mu^0 e_L + \bar{\mu}_L\gamma^\mu W_\mu^0\mu_L + \bar{\nu}_{\mu L}\gamma^\mu W_\mu^0\nu_{\mu L}) \\ &-2(\bar{e}_R\gamma^\mu W_\mu^0 e_R + \bar{\mu}_R\gamma^\mu W_\mu^0\mu_R)] \end{aligned} \tag{16}$$

Furthermore, defining

$$J_\mu(em) = -\bar{e}\gamma^\mu e - \bar{\mu}\gamma^\mu\mu \tag{17}$$

$$J_{\mu L} = -\bar{e}_L\gamma^\mu e_L - \bar{\mu}_L\gamma^\mu\mu_L + \bar{\nu}_{eL}\gamma^\mu\nu_{eL} + \bar{\nu}_{\mu L}\gamma^\mu\nu_{\mu L} \tag{18}$$

we can rewrite the interaction Lagrangian as

$$\begin{aligned}
L_{\text{int}} = {} & g' W_\mu^0 J_\mu(em) + \tfrac{1}{2}(gF_\mu - g' W_\mu^0) J_{\mu L} \\
& + \tfrac{1}{2} g C_\mu (\bar\nu_{eL}\gamma^\mu \nu_{eL} - \bar e_L \gamma^\mu e_L + \bar\mu_L \gamma^\mu \mu_L - \bar\nu_{\mu L}\gamma^\mu \nu_{\mu L}) \\
& + \tfrac{1}{2} g i D_\mu (\nu_{eL}\gamma^\mu \nu_{\mu L} + \bar e_L \gamma^\mu \mu_L - \bar\mu_L \gamma^\mu e_L - \bar\nu_{\mu L}\gamma^\mu \nu_{eL}) \\
& + \tfrac{1}{2} g E_\mu (\bar\nu_{eL}\gamma^\mu \nu_{\mu L} - \bar e_L \gamma^\mu \mu_L - \bar\mu_L \gamma^\mu e_L + \bar\nu_{\mu L}\gamma^\mu \nu_{eL}) \\
& + (1/\sqrt{2}) g U_\mu^+ (\bar\nu_{eL}\gamma^\mu e_L - \bar\nu_{\mu L}\gamma^\mu \mu_L) + \text{h.c.} \\
& + (1/\sqrt{2}) g V_\mu^+ (\bar\nu_{eL}\gamma^\mu \mu_L + \bar\nu_{\mu L}\gamma^\mu e_L) + \text{h.c.} \\
& + (1/\sqrt{2}) g W_\mu^+ (\bar\nu_{eL}\gamma^\mu e_L + \bar\nu_{\mu L}\gamma^\mu \mu_L) + \text{h.c.}
\end{aligned} \tag{19}$$

Now consider the interaction of the gauge fields U_μ^+ and V_μ^+ with the leptons. If the physical boson fields $\tilde U_\mu^+$ and $\tilde V_\mu^+$ are linear combinations defined by

$$\begin{aligned}
\tilde U_\mu^+ &= (\cos\theta) U_\mu^+ + (\sin\theta) V_\mu^+ \\
\tilde V_\mu^+ &= -(\sin\theta) U_\mu^+ + (\cos\theta) V_\mu^+
\end{aligned} \tag{20}$$

we can reexpress the last two terms of equation (19) containing U_μ^+ and V_μ^+ as

$$\begin{aligned}
& \tilde U_\mu^+[(\bar\nu_{eL}\cos\theta + \bar\nu_{\mu L}\sin\theta)\gamma^\mu e_L - (-\bar\nu_{eL}\sin\theta + \bar\nu_{\mu L}\cos\theta)\gamma^\mu \mu_L] \\
& \quad + \tilde V_\mu^+[(-\bar\nu_{eL}\sin\theta + \bar\nu_{\mu L}\cos\theta)\gamma^\mu e_L + (\bar\nu_{eL}\cos\theta + \bar\nu_{\mu L}\sin\theta)\gamma^\mu \mu_L]
\end{aligned} \tag{21}$$

The last two terms are forbidden, as they allow the possibility of flavor-changing charged current weak interactions. However, we keep these terms, as the lepton number change associated with these terms occurs at the instant of decay and not in the subsequent evolution of the neutrino state vector (Commins and Bucksbaum, 1983). We further define

$$\begin{aligned}
\nu_{e\theta} &= (\cos\theta)\nu_e + (\sin\theta)\nu_\mu \\
\nu_{\mu\theta} &= -(\sin\theta)\nu_e + (\cos\theta)\nu_\mu
\end{aligned} \tag{22}$$

Now, if $\tilde W_\mu^+$ is the physical gauge boson instead of W_μ^+, we may replace the last term in equation (19) by

$$(1/\sqrt{2}) g \tilde W_\mu^+ (\bar\nu_{eL\theta}\gamma^\mu e_L + \bar\nu_{\mu L\theta}\gamma^\mu \mu_L) + \text{h.c.} \tag{23}$$

In a similar way, if $\tilde C_\mu$ and $\tilde E_\mu$ are considered the physical gauge bosons, then the terms containing C_μ and E_μ can be respectively replaced by

$$\begin{aligned}
& \tfrac{1}{2} g \tilde C_\mu (\bar\nu_{eL\theta}\gamma^\mu \nu_{eL\theta} - \bar e_L \gamma^\mu e_L + \bar\mu_L \gamma^\mu e_L + \bar\nu_{\mu L\theta}\gamma^\mu \nu_{eL\theta}) \\
& \quad + \tfrac{1}{2} g \tilde E_\mu (\bar\nu_{eL\theta}\gamma^\mu \nu_{eL\theta} - \bar e_L \gamma^\mu \mu_L - \bar\mu_L \gamma^\mu e_L + \bar\nu_{\mu L\theta}\gamma^\mu \nu_{eL\theta})
\end{aligned}$$

The term containing D_μ remains invariant under replacement $\nu_e \to \nu_{e\theta}$, $\nu_\mu \to \nu_{\mu\theta}$. Regarding F_μ and W_μ^0 as the physical gauge bosons, one redefines equations (17) and (18) with ν_e, ν_μ replaced by $\nu_{e\theta}$ and $\nu_{\mu\theta}$, respectively.

4. THE NEUTRINO OSCILLATIONS

In view of our present work for neutrino oscillations in matter, we summarize here work done (e.g., Commins and Bucksbaum, 1983) on neutrino oscillations in vacuum. Consider a neutrino formed in the state $|\nu_{e\theta}\rangle$ at a time $t=0$ and if $|\nu_e\rangle$ and $|\nu_\mu\rangle$ are assumed to have time evolutions

$$\begin{aligned} |\nu_e(t)\rangle &= \exp(-iE_{\nu_e}t/\hbar)|\nu_e(0)\rangle \\ |\nu_\mu(t)\rangle &= \exp(-iE_{\nu_\mu}t/\hbar)|\nu_\mu(0)\rangle \end{aligned} \tag{24}$$

where

$$\begin{aligned} E_{\nu_e} &= (p_{\nu_e}^2 + m_{\nu_e}^2)^{1/2} = (p^2 + m_{\nu_e}^2)^{1/2} \\ E_{\nu_\mu} &= (p_{\nu_\mu}^2 + m_{\nu_\mu}^2)^{1/2} = (p^2 + m_{\nu_\mu}^2)^{1/2} \end{aligned}$$

Also, we have assumed

$$p_{\nu_e} = p_{\nu_\mu} = p$$

Equation (22) evolves to

$$\begin{aligned} |\nu_{e\theta}(t)\rangle = {} & \exp(-iE_{\nu_e}t/\hbar)|\nu_e(0)\rangle \cos\theta \\ & + \exp(-iE_{\nu_\mu}t/\hbar)|\nu_\mu(0)\rangle \sin\theta \end{aligned} \tag{25}$$

This can be written as

$$\begin{aligned} |\nu_{e\theta}(t) = {} & \{[\exp(-iE_{\nu_e}t/\hbar)\cos^2\theta + \exp(-iE_{\nu_\mu}t/\hbar)\sin^2\theta]|\nu_{e\theta}(0)\rangle \\ & + (\cos\theta\sin\theta)[\exp(-iE_{\nu_\mu}t/\hbar) - \exp(-iE_{\nu_e}t/\hbar)]|\nu_{\mu\theta}(0)\rangle \end{aligned} \tag{26}$$

The probability for a neutrino originally in the state $|\nu_e\rangle$ to be in the state $|\nu_\mu\rangle$ at time t is given by

$$\begin{aligned} & P(\nu_e \leftrightarrow \nu_\mu) \\ & = |\langle\nu_\mu|\nu_e(t)|^2 \\ & = (\cos^2\theta\sin^2\theta)|\exp(-iE_{\nu_\mu}t/\hbar) - \exp(-iE_{\nu_e}t/\hbar)|^2 \\ & = \tfrac{1}{2}(\sin^2 2\theta)[1 - \cos(E_{\nu_\mu} - E_{\nu_e})t/\hbar] \end{aligned} \tag{27}$$

If a neutrino beam travels a distance R in time t, putting $R = Ct$ and assuming $p \gg m_{\nu_e}$, m_{ν_μ}, we can rewrite equation (27) as

$$P(\nu_e \leftrightarrow \nu_\mu) = \tfrac{1}{2}(\sin^2 2\theta)\left(1 - \cos\frac{m_{\nu_\mu}^2 - m_{\nu_e}^2}{2p}\frac{C^2}{\hbar}R\right) \tag{28}$$

The neutrino oscillation length is defined by the equation

$$\frac{m^2_{\nu_\mu}-m^2_{\nu_e}}{2p}\frac{C^2}{\hbar}R=\frac{2\pi R}{L}$$

where L is the oscillation length

$$L=4\pi\hbar p/(m^2_{\nu_\mu}-m^2_{\nu_e})C^2 \tag{29}$$

In view of the fact that usually data on three neutrino flavors are analyzed supposing that the dominant part comes only from one pair of neutrinos (i.e., coming back to the case of two neutrino flavors), even a detailed calculation for the dominant contribution would lead to the same result as given in equations (28) and (29).

5. NEUTRINO OSCILLATION IN MATTER

In realistic situations neutrinos pass through matter. It would be interesting to consider the possible effects of matter on the neutrino oscillations.

For a neutrino passing through matter in the Z direction, if its state $\nu(0,0)$ at $t=0$ and $Z=0$ is known, then its state $\nu(t,Z)$ at finite t and Z can be written as

$$\nu(t,Z)=\nu(0,0)\exp(-iE_\nu t|\hbar)\exp(iKn_\nu Z) \tag{30}$$

where n_ν is the refractive index of the scattering substance and is given by (Commins and Bucksbaum, 1983)

$$n_\nu=1+(2\pi N/K^2)f_\nu(0) \tag{31}$$

where N is the number of identical scattering centers per unit volume and $\bar{K}$ is the wave vector in the direction of the scattered wave ($\bar{K}=\bar{K}_0$ is the incident wave vector). $f_\nu(0)$ is the forward scattering amplitude.

Thus, using equation (30), one can recast equation (22) as follows:

$$\begin{aligned}\nu_{e\theta}(Z,t)=&\cos\theta\exp(iKn_{\nu_e}Z)\exp(-iE_{\nu_e}t|\hbar)\nu_e\\&+\sin\theta\exp(iKn_{\nu_\mu}Z)\exp(-iE_{\nu_\mu}t|\hbar)\nu_\mu\end{aligned} \tag{32a}$$

$$\begin{aligned}\nu_{\mu\theta}(Z,t)=&-\sin\theta\exp(iKn_{\nu_e}Z)\exp(-iE_{\nu_e}t|\hbar)\nu_e\\&+\cos\theta\exp(iKn_{\nu_\mu}Z)\exp(-iE_{\nu_\mu}t|\hbar)\nu_\mu\end{aligned} \tag{32b}$$

Furthermore, reemploying equation (22), one obtains from equation (32a) the following relation:

$$\begin{aligned}|\nu_{e\theta}(Z,t)\rangle=&[\exp(iKn_{\nu_e}Z)\exp(-iE_{\nu_e}t|\hbar)\cos^2\theta\\&+\exp(iKn_{\nu_\mu}Z)\exp(-iE_{\nu_\mu}t|\hbar)\sin^2\theta]|\nu_{e\theta}\rangle\\&+(\sin\theta\cos\theta)[\exp(iKn_{\nu_\mu}Z)\exp(-iE_{\nu_\mu}t|\hbar)\\&-\exp(iKn_{\nu_e}Z)\exp(-iE_{\nu_e}t|\hbar)]|\nu_{\mu\theta}\rangle\end{aligned} \tag{33}$$

The probability for a neutrino originally in the state $\nu_{e\theta}(t, Z)$ at time t and at the point Z to be in the state $|\nu_{\mu\theta}\rangle$ is given by

$$|\langle \nu_{\mu\theta}|\nu_{e\theta}(z, t)\rangle|^2 = \tfrac{1}{2}(\sin^2 2\theta)\{1 - \cos[(E_{\nu_\mu} - E_{\nu_e})t|\hbar + (n_{\nu_e} - n_{\nu_\mu})|KZ\rangle]\} \quad (34)$$

We observe that the argument of the cosine function consists of a part dependent on time and a part dependent on space. For vacuum the refractive indices $n_{\nu_e} = n_{\nu_\mu} = 1$ and hence the time part survives. This reduces to the case where the medium is vacuum.

It is interesting to note that even if $m_{\nu_\mu} = m_{\nu_e}$ or $m_{\nu_\mu} = m_{\nu_e} = 0$, neutrino oscillations exist and in this case the oscillation length l'_m is given by

$$2\pi / l'_m = (n_{\nu_e} - n_{\nu_\mu})K$$

as already noted by Wolfenstein.

6. DISCUSSION OF RESULTS

In this work we consider only two lepton generations and specifically discuss ν_e and ν_μ oscillations in matter. Oscillations in matter among other neutrino flavors can be similarly discussed, as is now rather trivial. However, we have not yet attempted our technique in detail with three lepton generations. In this work it is observed that even if the neutrinos ν_e and ν_μ have the same mass or are massless the neutrino oscillations do exist. This result was already noted by Wolfenstein.

It may also be noted that *CPT* invariance implies the following relation for the corresponding antineutrino oscillations:

$$P(\bar{\nu}_\mu \leftrightarrow \bar{\nu}_e) = P(\nu_e \leftrightarrow \nu_\mu)$$

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